Test code: ME I/ME II, 2009

Syllabus for ME I, 2009

Matrix Algebra: Matrices and Vectors, Matrix Operations.

Permutation and Combination.

Calculus: Functions, Limits, Continuity, Differentiation of functions of one or more variables, Unconstrained optimization, Definite and Indefinite Integrals: integration by parts and integration by substitution, Constrained optimization of functions of not more than two variables.

Algebra: Binomial Theorem, AP, GP, HP, exponential, logarithmic series.

Theory of Polynomial Equations (up to third degree).

Elementary Statistics: Measures of central tendency; dispersion, correlation and regression, Elementary probability theory, Probability distributions.

Sample Questions for ME I (Mathematics), 2009

1. An infinite geometric series has first term 1 and sum 4. Its common ratio is

   A $\frac{1}{2}$

   B $\frac{3}{4}$

   C 1

   D $\frac{1}{3}$

2. A continuous random variable X has a probability density function $f(x) = 3x^2$ with $0 \leq x \leq 1$. If $P(X \leq a) = P(x > a)$, then a is:

   A $\frac{1}{\sqrt{6}}$

   B $\left(\frac{1}{3}\right)^{\frac{1}{2}}$

   C $\frac{1}{2}$

   D $\left(\frac{1}{2}\right)^{\frac{1}{3}}$

3. If $f(x) = \sqrt{e^x + \sqrt{e^x + \sqrt{e^x + \ldots}}}$, then $f'(x)$ equals to

   A $\frac{f(x)-1}{2f(x)+1}$.

   B $\frac{f^2(x)-f(x)}{2f(x)-1}$.

   C $\frac{2f(x)+1}{f^2(x)+f(x)}$.

   D $\frac{f(x)}{2f(x)+1}$.

4. $\lim_{x \to 4} \frac{\sqrt{x+5}-3}{x-4}$ is

   A $\frac{1}{6}$

   B 0

   C $\frac{1}{4}$

   D not well defined

5. If $X = 2^{65}$ and $Y = 2^{64} + 2^{63} + \ldots + 2^1 + 2^0$, then

   A $Y = X + 2^{64}$.

B $X = Y$.

C $Y = X + 1$.

D $Y = X - 1$.

6. $\int_0^1 \frac{e^x}{e^x+1} dx =$

A $log(1 + e)$.

B $log2$.

C $log\frac{1+e}{2}$.

D $2log(1 + e)$.

7. There is a box with ten balls. Each ball has a number between 1 and 10 written on it. No two balls have the same number. Two balls are drawn (simultaneously) at random from the box. What is the probability of choosing two balls with odd numbers?

A $\frac{1}{9}$.

B $\frac{1}{2}$.

C $\frac{2}{9}$.

D $\frac{1}{3}$.

8. A box contains 100 balls. Some of them are white and the remaining are red. Let X and Y denote the number of white and red balls respectively. The correlation between X and Y is

A 0.

B 1.

C -1.

D some real number between $-\frac{1}{2}$ and $\frac{1}{2}$.

9. Let f, g and h be real valued functions defined as follows: $f(x) = x(1 - x)$, $g(x) = \frac{x}{2}$ and $h(x) = \min\{f(x), g(x)\}$ with $0 \leq x \leq 1$. Then h is

A continuous and differentiable

B is differentiable but not continuous

C is continuous but not differentiable

D is neither continuous nor differentiable

10. In how many ways can three persons, each throwing a single die once, make a score of 8?

A 5

B 15

C 21

D 30

11. If $f(x)$ is a real valued function such that

$$2f(x) + 3f(-x) = 55 - 7x,$$

for every $x \in \Re$, then $f(3)$ equals

A 40

B 32

C 26

D 10

12. Two persons, A and B, make an appointment to meet at the train station between 4 P.M. and 5 P.M.. They agree that each is to wait not more than 15 minutes for the other. Assuming that each is independently equally likely to arrive at any point during the hour, find the probability that they meet.

A $\frac{15}{16}$

B $\frac{7}{16}$

C $\frac{5}{24}$

D $\frac{22}{175}$

13. If x_1, x_2, x_3 are positive real numbers, then

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_1}$$

is always

A ≤ 3

B $\leq 3^{\frac{1}{3}}$

C ≥ 3

D 3

14. $\lim_{n\to\infty} \frac{1^2+2^2+\ldots+n^2}{n^3}$ equals

A 0

B $\frac{1}{3}$

C $\frac{1}{6}$

D 1.

15. Suppose b is an odd integer and the following two polynomial equations have a common root.

$$x^2 - 7x + 12 = 0$$
$$x^2 - 8x + b = 0.$$

The root of $x^2 - 8x + b = 0$ that is not a root of $x^2 - 7x + 12 = 0$ is

A 2

B 3

C 4

D 5

16. Suppose $n \geq 9$ is an integer. Let $\mu = n^{\frac{1}{2}} + n^{\frac{1}{3}} + n^{\frac{1}{4}}$. Then, which of the following relationships between n and μ is correct?

A $n = \mu$.

B $n > \mu$.

C $n < \mu$.

D None of the above.

17. Which of the following functions $f : \Re \to \Re$ satisfies the relation $f(x+y) = f(x) + f(y)$?

A $f(z) = z^2$

B $f(z) = az$ for some real number a

C $f(z) = \log z$

D $f(z) = e^z$

18. For what value of a does the following equation have a unique solution?

$$\begin{vmatrix} x & a & 2 \\ 2 & x & 0 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

A 0

B 1

C 2

D 4

19. Let

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

where l, m, n, a, b, c are non-zero numbers. Then $\frac{dy}{dx}$ equals

A

$$\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

B

$$\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix}$$

C

$$\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

D

$$\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l-a & m-b & n-c \\ 1 & 1 & 1 \end{vmatrix}$$

20. If $f(x) = |x-1| + |x-2| + |x-3|$, then $f(x)$ is differentiable at

A 0

B 1

C 2

D 3

21. If $(x-a)^2+(y-b)^2=c^2$, then $1+\left[\frac{dy}{dx}\right]^2$ is independent of

A a

B b

C c

D Both b and c.

22. A student is browsing in a second-hand bookshop and finds n books of interest. The shop has m copies of each of these n books. Assuming he never wants duplicate copies of any book, and that he selects at least one book, how many ways can he make a selection? For example, if there is one book of interest with two copies, then he can make a selection in 2 ways.

A $(m+1)^n-1$

B nm

C $2^{nm}-1$

D $\frac{nm!}{(m!)(nm-m)!}-1$

23. Determine all values of the constants A and B such that the following function is continuous for all values of x.

$$f(x)=\begin{cases} Ax-B & \text{if } x\leq -1 \\ 2x^2+3Ax+B & \text{if } -1<x\leq 1 \\ 4 & \text{if } x>1 \end{cases}$$

A $A=B=0$

B $A=\frac{3}{4}$, $B=-\frac{1}{4}$

C $A=\frac{1}{4}$, $B=\frac{3}{4}$

D $A=\frac{1}{2}$, $B=\frac{1}{2}$

24. The value of $\lim_{x\to\infty}(3^x+3^{2x})^{\frac{1}{x}}$ is

A 0

B 1

C e

D 9

25. A computer while calculating correlation coefficient between two random variables X and Y from 25 pairs of observations obtained the following results: $\sum X = 125$, $\sum X^2 = 650$, $\sum Y = 100$, $\sum Y^2 = 460$, $\sum XY = 508$. It was later discovered that at the time of inputing, the pair $(X = 8, Y = 12)$ had been wrongly input as $(X = 6, Y = 14)$ and the pair $(X = 6, Y = 8)$ had been wrongly input as $(X = 8, Y = 6)$. Calculate the value of the correlation coefficient with the correct data.

A $\frac{4}{5}$

B $\frac{2}{3}$

C 1

D $\frac{5}{6}$

26. The point on the curve $y = x^2 - 1$ which is nearest to the point $(2, -0.5)$ is

A $(1, 0)$

B $(2, 3)$

C $(0, -1)$

D None of the above

27. If a probability density function of a random variable X is given by $f(x) = kx(2 - x), 0 \leq x \leq 2$, then mean of X is

A $\frac{1}{2}$

B 1

C $\frac{1}{5}$

D $\frac{3}{4}$

28. Suppose X is the set of all integers greater than or equal to 8. Let $f : X \to \Re$. and $f(x + y) = f(xy)$ for all $x, y \geq 4$. If $f(8) = 9$, then $f(9) =$

A 8.

B 9.

C 64.

D 81.

29. Let $f : \Re \to \Re$ be defined by $f(x) = (x-1)(x-2)(x-3)$. Which of the following is true about f?

A It decreases on the interval $[2 - 3^{-\frac{1}{2}}, 2 + 3^{-\frac{1}{2}}]$

B It increases on the interval $[2 - 3^{-\frac{1}{2}}, 2 + 3^{-\frac{1}{2}}]$

C It decreases on the interval $(-\infty, 2 - 3^{-\frac{1}{2}}]$

D It decreases on the interval $[2, 3]$

30. A box with no top is to be made from a rectangular sheet of cardboard measuring 8 metres by 5 metres by cutting squares of side x metres out of each corner and folding up the sides. The largest possible volume in cubic metres of such a box is

A 15

B 12

C 20

D 18

Syllabus for ME II (Economics), 2009

Microeconomics: Theory of consumer behaviour, Theory of Production, Market Structures under Perfect Competition, Monopoly, Price Discrimination, Duopoly with Cournot and Bertrand Competition (elementary problems) and Welfare economics.

Macroeconomics: National Income Accounting, Simple Keynesian Model of Income Determination and the Multiplier, IS-LM Model, Model of Aggregate Demand and Aggregate Supply, Harrod-Domar and Solow Models of Growth, Money, Banking and Inflation.

Sample questions for ME II (Economics), 2009

1. Consider the following model of the economy:

$$\begin{aligned} C &= c_0 + c_1 Y_D \\ T &= t_0 + t_1 Y \\ Y_D &= Y - T. \end{aligned}$$

C denotes consumption, $c_0 > 0$ denotes autonomous consumption, $0 < c_1 < 1$ is the marginal propensity to consume, Y, denotes income, T denotes taxes, Y_D denotes disposable income and $t_0 > 0$, $t_1 > 0$. Assume a closed economy where government spending G, and investment I, are exogenously given by $\bar{G}$ and $\bar{I}$ respectively.

- (i) Interpret t_1 in words. Is it greater or less than 1? Explain your answer.
- (ii) Solve for equilibrium output, Y^*.
- (iii) What is the multiplier ? Does the economy respond more to changes in autonomous spending (such as changes in $c_0, \bar{G}$, and, $\bar{I}$) when t_1 is zero or when t_1 is positive ? Explain.

[5]+[5]+[10]

2. Consider an agent who values consumption in periods 0 and 1 according to the utility function

$$u(c_0, c_1) = \log c_0 + \delta \log c_1$$

where $0 < \delta < 1$. Suppose that the agent has wealth ω in period 0 of which she can save any portion in order to consume in period 1. If she saves Re. 1, she is paid interest r so that her budget constraint is

$$c_0 + \tfrac{c_1}{1+r} = \omega$$

- (i) Derive the agent's demand for c_0 and c_1 as a function of r and ω.
- (ii) What happens to c_0 and c_1 as r increases? Interpret.
- (iii) For what relationship between ω and r will she consume the same amount in both periods?

  [8]+[6]+[6]

3. Consider a firm with production function $F(x_1, x_2) = \min(2x_1, x_1 + x_2)$ where x_1 and x_2 are amounts of factors 1 and 2.

- (i) Draw an isoquant for output level 10.

- (ii) Show that the production function exhibits constant returns to scale.
- (iii) Suppose that the firm faces input prices $w_1 = w_2 = 1$. What is the firms' cost function?

  [8]+[6]+[6]

4. Consider an exchange economy consisting of two individuals 1 and 2, and two goods X and Y. The utility function of individual i, $U_i = X_i + Y_i$. Individual 1 has 3 units of X and 7 units of Y to begin with. Similarly, individual 2 has 7 units of X and 3 units of Y to begin with.

- (i) What is the set of Pareto optimal outcomes in this economy? Justify your answer.
- (ii) What is the set of perfectly competitive (Walrasian) outcomes? You may use diagrams for parts (i) and (ii).
- (iii) Are the perfectly competitive outcomes Pareto optimal? Does this result hold generally in all exchange economies?

  [8]+[8]+[4]

5. A monopoly sells its product in two separate markets. The inverse demand function in market 1 is given by $q_1 = 10 - p_1$, and the inverse demand function in market 2 is given by $q_2 = a - p_2$, where $10 < a \leq 20$. The monopolist's cost function is $C(q) = 5q$, where q is aggregate output.

- (i) Suppose the monopolist must set the same price in both markets. What is its optimal price? What is the reason behind the restriction that $a \leq 20$?
- (ii) Suppose the monopolist can charge different prices in the two markets. Compute the prices it will set in the two markets.
- (iii) Under what conditions does the monopolist benefit from the ability to charge different prices?
- (iv) Compute consumers' surplus in cases (i) and (ii). Who benefits from differential pricing and who does not relative to the case where the same price is charged in both markets?

[5]+[5]+[5]+[5]

6. Consider an industry with 3 firms, each having marginal cost equal to 0. The inverse demand curve facing this industry is $p = 120 - q$, where q is aggregate output.

- (i) If each firm behaves as in the Cournot model, what is firm 1's optimal output choice as a function of its beliefs about other firms' output choices?
- (ii) What output do the firms produce in equilibrium?
- (iii) Firms 2 and 3 decide to merge and form a single firm with marginal cost still equal to 0. What output do the two firms produce in equilibrium? Is firm 1 better off as a result? Are firms 2 and 3 better off post-merger? Would it be better for all the firms to form a cartel instead? Explain in each case.

[3]+[5]+[12]

7. Suppose the economy's production function is given by

$$Y_t = 0.5\sqrt{K_t}\sqrt{N_t} \tag{1}$$

Y_t denotes output, K_t denotes the aggregate capital stock in the economy, and N denotes the number of workers (which is fixed). The evolution of the capital stock is given by,

$$K_{t+1} = sY_t + (1-\delta)K_t \tag{2}$$

where the savings rate of the economy is denoted by, s, and the depreciation rate is given by, δ.

- (i) Using equation (2), show that the change in the capital stock per worker, $\frac{K_{t+1}-K_t}{N}$, is equal to savings per worker minus depreciation per worker.
- (ii) Derive the economy's steady state levels of $\frac{K}{N}$ and $\frac{Y}{N}$ in terms of the savings rate and the depreciation rate.
- (iii) Derive the equation for the steady state level of consumption per worker in terms of the savings rate and the depreciation rate.
- (iv) Is there a savings rate that is optimal, i.e., maximizes steady state consumption per worker ? If so, derive an expression for the optimal savings rate. Using words and graphs, discuss your answer.

[2]+[6]+[6]+[6]

8. Suppose there are 10 individuals in a society, 5 of whom are of high ability, and 5 of low ability. Individuals know their own abilities. Suppose that each individual lives for two periods and is deciding whether or not to go to college in period 1. When individuals make decisions in period 1, they choose that option which gives the highest lifetime payoff, i.e., the sum of earnings and expenses in both periods.

Education can only be acquired in period 1. In the absence of schooling, high and low ability individuals can earn y_H and y_L respectively in each period.

With education, period 2 earning increases to $(1 + a)y_H$ for high ability types and $(1 + a)y_L$ for low ability types. Earnings would equal 0 in period 1 if an individual decided to go to college in that period. Tuition fee for any individual is equal to T. Assume y_H and y_L are both positive, as is T.

- (i) Find the condition that determines whether each type of person will go to college in period 1. What is the minimum that a can be if it is to be feasible for any type of individual to acquire education?
- (ii) Suppose $y_H = 50$, $y_L = 40$, $a = 3$. For what values of T will a high ability person go to college? And a low ability person? Which type is more likely to acquire education?
- (iii) Now assume the government chooses to subsidise education by setting tuition equal to 60. What happens to educational attainment?
- (iv) Suppose now to pay for the education subsidy, the government decides to impose a $x\%$ tax on earnings in any period greater than 50. So if an individual earns 80 in a period, he would pay a tax in that period equal to $x\%$ of 30.The government wants all individuals to acquire education, and also wants to cover the cost of the education subsidy in period 1 through tax revenues collected in both periods. What value of x should the government set?

[5]+[5]+[2]+[8]

9. Consider the goods market with exogenous (constant) investment $\bar{I}$, exogenous government spending, $\bar{G}$ and constant taxes, T. The consumption equation is given by,

$$C = c_0 + c_1(Y - T),$$

where C denotes consumption, c_0 denotes autonomous consumption, and c_1 the marginal propensity to consume.

- (i) Solve for equilibrium output. What is the value of the multiplier ?
- (ii) Now let investment depend on Y and the interest rate, i

  $$I = b_0 + b_1 Y - b_2 i,$$

  where b_0 and b_1 are parameters. Solve for equilibrium output. At a given interest rate, is the effect of an increase in autonomous spending bigger than it was in part (i)? In answering this, assume that $c_1 + b_1 < 1$.
- (iii) Now, introduce the financial market equilibrium condition

  $$\frac{M}{P} = d_1 Y - d_2 i,$$

  where $\frac{M}{P}$ denotes the real money supply. Derive the multiplier. Assume that investment is given by the equation in part (ii).

- (iv) Is the multiplier you obtained in part (iii) smaller or larger than the multiplier you obtained in part (i). Explain how your answer depends on the behavioral equations for consumption, investment, and money demand.

[5]+[5]+[5]+[5]

10 (i) A college is trying to fill one remaining seat in its Masters programme. It judges the merit of any applicant by giving him an entrance test. It is known that there are two interested applicants who will apply sequentially. If the college admits the first applicant, it cannot admit the second. If it rejects the first applicant, it must admit the second. It is not possible to delay a decision on the first applicant till the second applicant is tested. At the time of admitting or rejecting the first applicant, the college thinks the second applicant's mark will be a continuous random variable drawn from the uniform distribution between 0 and 100. (Recall that a random variable x is uniformly distibuted on $[a, b]$ if the density function of x is given by $f(x) = \frac{1}{b-a}$ for $x \in [a, b]$). If the college wants to maximize the expected mark of its admitted student, what is the lowest mark for which it should admit the first applicant?

(ii) Now suppose there are three applicants who apply sequentially. Before an applicant is tested, it is known that his likely mark is an independent continuous random variable drawn from the uniform distribution between 0 and 100. What is the lowest mark for which the college should admit the first student? What is the lowest mark for which the college should admit the second student in case the first is rejected?
[8]+[12]