Test code: ME I/ME II, 2010

Syllabus for ME I, 2010

Matrix Algebra: Matrices and Vectors, Matrix Operations.

Permutation and Combination.

Calculus: Functions, Limits, Continuity, Differentiation of functions of one or more variables, Unconstrained optimization, Definite and Indefinite Integrals: integration by parts and integration by substitution, Constrained optimization of functions of not more than two variables.

Algebra: Binomial Theorem, AP, GP, HP, exponential, logarithmic series.

Theory of Polynomial Equations (up to third degree).

Elementary Statistics: Measures of central tendency; dispersion, correlation and regression, Elementary probability theory, Probability distributions.

Sample Questions for ME I (Mathematics), 2010

1. The value of $100\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{99.100}\right]$
   (a) is 99,
   (b) is 100,
   (c) is 101,
   (d) is $\frac{(100)^2}{99}$.

2. The function $f(x) = x(\sqrt{x}+\sqrt{x+9})$ is
   (a) continuously differentiable at $x = 0$,
   (b) continuous but not differentiable at $x = 0$,
   (c) differentiable but the derivative is not continuous at $x = 0$,
   (d) not differentiable at $x = 0$.

3. Consider a GP series whose first term is 1 and the common ratio is a positive integer $r(> 1)$. Consider an AP series whose first term is 1 and whose $(r+2)^{\text{th}}$ term coincides with the third term of the GP series. Then the common difference of the AP series is
   (a) $r - 1$,
   (b) r,
   (c) $r + 1$,
   (d) $r + 2$.

4. The first three terms of the binomial expansion $(1 + x)^n$ are $1, -9, \frac{297}{8}$ respectively. What is the value of n?
   (a) 5
   (b) 8
   (c) 10
   (d) 12

5. Given $log_p x = \alpha$ and $log_q x = \beta$, the value of $log_{\frac{p}{q}} x$ equals
   (a) $\frac{\alpha\beta}{\beta-\alpha}$,
   (b) $\frac{\beta-\alpha}{\alpha\beta}$,
   (c) $\frac{\alpha-\beta}{\alpha\beta}$,
   (d) $\frac{\alpha\beta}{\alpha-\beta}$.

6. Let $P = \{1, 2, 3, 4, 5\}$ and $Q = \{1, 2\}$. The total number of subsets X of P such that $X \cap Q = \{2\}$ is

   (a) 6,

   (b) 7,

   (c) 8,

   (d) 9.

7. An unbiased coin is tossed until a head appears. The expected number of tosses required is

   (a) 1,

   (b) 2,

   (c) 4,

   (d) ∞.

8. Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{c}{x^2} & \text{if } x \geq c \\ 0 & x < c. \end{cases}$$

   Then the expectation of X is

   (a) 0,

   (b) ∞,

   (c) $\frac{1}{c}$,

   (d) $\frac{1}{c^2}$.

9. The number of real solutions of the equation $x^2 - 5|x| + 4 = 0$ is

   (a) two,

   (b) three,

   (c) four.

   (d) None of these.

10. Range of the function $f(x) = \frac{x^2}{1+x^2}$ is

   (a) $[0, 1)$,

   (b) $(0, 1)$,

   (c) $[0, 1]$.

   (d) $(0, 1]$.

11. If a, b, c are in AP, then the value of the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

is

(a) $b^2 - 4ac$,

(b) $ab + bc + ca$,

(c) $2b - a - c$,

(d) $3b + a + c$.

12. If $a < b < c < d$, then the equation $(x-a)(x-b) + 2(x-c)(x-d) = 0$ has

(a) both the roots in the interval $[a, b]$,

(b) both the roots in the interval $[c, d]$,

(c) one root in the interval (a, b) and the other root in the interval (c, d),

(d) one root in the interval $[a, b]$ and the other root in the interval $[c, d]$.

13. Let f and g be two differentiable functions on $(0, 1)$ such that $f(0) = 2$, $f(1) = 6$, $g(0) = 0$ and $g(1) = 2$. Then there exists $\theta \in (0, 1)$ such that $f'(\theta)$ equals

(a) $\frac{1}{2}g'(\theta)$,

(b) $2g'(\theta)$,

(c) $6g'(\theta)$,

(d) $\frac{1}{6}g'(\theta)$.

14. The minimum value of $log_x a + log_a x$, for $1 < a < x$, is

(a) less than 1,

(b) greater than 2,

(c) greater than 1 but less than 2.

(d) None of these.

15. The value of $\int\limits_4^9 \frac{1}{2x(1+\sqrt{x})} dx$ equals

(a) $log_e 3 - log_e 2$,

(b) $2log_e 2 - log_e 3$,

(c) $2log_e 3 - 3log_e 2$,

(d) $3log_e 3 - 2log_e 2$.

16. The inverse of the function $f(x) = \frac{1}{1+x}$, $x > 0$, is

(a) $(1 + x)$,
(b) $\frac{1+x}{x}$,
(c) $\frac{1-x}{x}$,
(d) $\frac{x}{1+x}$.

17. Let X_i, $i = 1, 2, \ldots, n$ be identically distributed with variance σ^2. Let $cov(X_i, X_j) = \rho$ for all $i \neq j$. Define $\bar{X}_n = \frac{1}{n} \sum X_i$ and let $a_n = Var(\bar{X}_n)$. Then $\lim_{n\to\infty} a_n$ equals

(a) 0,
(b) ρ,
(c) $\sigma^2 + \rho$,
(d) $\sigma^2 + \rho^2$.

18. Let X be a Normally distributed random variable with mean 0 and variance 1. Let $\Phi(.)$ be the cumulative distribution function of the variable X. Then the expectation of $\Phi(X)$ is

(a) $-\frac{1}{2}$,
(b) 0,
(c) $\frac{1}{2}$,
(d) 1.

19. Consider any finite integer $r \geq 2$. Then $\lim_{x\to 0} \left[\frac{log_e \left(\sum_{k=0}^{r} x^k \right)}{\left(\sum_{k=1}^{\infty} \frac{x^k}{k!} \right)} \right]$ equals

(a) 0,
(b) 1,
(c) e,
(d) $log_e 2$.

20. Consider 5 boxes, each containing 6 balls labelled $1, 2, 3, 4, 5, 6$. Suppose one ball is drawn from each of the boxes. Denote by b_i, the label of the ball drawn from the i-th box, $i = 1, 2, 3, 4, 5$. Then the number of ways in which the balls can be chosen such that $b_1 < b_2 < b_3 < b_4 < b_5$ is

(a) 1,
(b) 2,
(c) 5,
(d) 6.

21. The sum $\sum_{r=0}^{m} \binom{n+r}{r}$ equals

(a) $\binom{n+m+1}{n+m}$,

(b) $(n+m+1)\binom{n+m}{n+1}$,

(c) $\binom{n+m+1}{n}$,

(d) $\binom{n+m+1}{n+1}$.

22. Consider the following 2-variable linear regression where the error ϵ_i's are independently and identically distributed with mean 0 and variance 1;

$$y_i = \alpha + \beta(x_i - \bar{x}) + \epsilon_i, \qquad i = 1, 2, \ldots, n.$$

Let $\hat{\alpha}$ and $\hat{\beta}$ be ordinary least squares estimates of α and β respectively. Then the correlation coefficient between $\hat{\alpha}$ and $\hat{\beta}$ is

(a) 1,

(b) 0,

(c) -1,

(d) $\frac{1}{2}$.

23. Let f be a real valued continuous function on $[0, 3]$. Suppose that $f(x)$ takes only rational values and $f(1) = 1$. Then $f(2)$ equals

(a) 2,

(b) 4,

(c) 8.

(d) None of these.

24. Consider the function $f(x_1, x_2) = \int_0^{\sqrt{x_1^2+x_2^2}} e^{-(w^2/(x_1^2+x_2^2))} dw$ with the property that $f(0, 0) = 0$. Then the function $f(x_1, x_2)$ is

(a) homogeneous of degree -1,

(b) homogeneous of degree $\frac{1}{2}$,

(c) homogeneous of degree 1.

(d) None of these.

25. If $f(1) = 0$, $f'(x) > f(x)$ for all $x > 1$, then $f(x)$ is

(a) positive valued for all $x > 1$,

(b) negative valued for all $x > 1$,

(c) positive valued on $(1, 2)$ but negative valued on $[2, \infty)$.

(d) None of these.

26. Consider the constrained optimization problem

$$\max_{x\geq 0, y\geq 0} (ax+by) \text{ subject to } (cx+dy)\leq 100$$

where a, b, c, d are positive real numbers such that $\frac{d}{b} > \frac{(c+d)}{(a+b)}$. The unique solution (x^*, y^*) to this constrained optimization problem is

(a) $(x^* = \frac{100}{a}, y^* = 0)$,
(b) $(x^* = \frac{100}{c}, y^* = 0)$,
(c) $(x^* = 0, y^* = \frac{100}{b})$,
(d) $(x^* = 0, y^* = \frac{100}{d})$.

27. For any real number x, let $[x]$ be the largest integer not exceeding x. The domain of definition of the function $f(x) = \left(\sqrt{|[|x|-2]|-3}\right)^{-1}$ is

(a) $[-6, 6]$,
(b) $(-\infty, -6) \cup (+6, \infty)$,
(c) $(-\infty, -6] \cup [+6, \infty)$,
(d) None of these.

28. Let $f : \Re \to \Re$ and $g : \Re \to \Re$ be defined as

$$f(x) = \begin{cases} -1 & \text{if } x < -\frac{1}{2} \\ -\frac{1}{2} & \text{if } -\frac{1}{2} \leq x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

and $g(x) = 1 + x - [x]$, where $[x]$ is the largest integer not exceeding x. Then $f(g(x))$ equals

(a) -1,
(b) $-\frac{1}{2}$,
(c) 0,
(d) 1.

29. If f is a real valued function and $a_1 f(x) + a_2 f(-x) = b_1 - b_2 x$ for all x with $a_1 \neq a_2$ and $b_2 \neq 0$. Then $f(\frac{b_1}{b_2})$ equals

(a) 0,
(b) $-\left(\frac{2a_2b_1}{a_1^2 - a_2^2}\right)$,
(c) $\frac{2a_2b_1}{a_1^2 - a_2^2}$.
(d) More information is required to find the exact value of $f(\frac{b_1}{b_2})$.

30. For all $x, y \in (0, \infty)$, a function $f : (0, \infty) \to \Re$ satisfies the inequality

$$|f(x) - f(y)| \leq |x - y|^3.$$

Then f is

(a) an increasing function,

(b) a decreasing function,

(c) a constant function.

(d) None of these.

Syllabus for ME II (Economics), 2010

Microeconomics: Theory of consumer behaviour, Theory of Production, Market Structures under Perfect Competition, Monopoly, Price Discrimination, Duopoly with Cournot and Bertrand Competition (elementary problems) and Welfare economics.

Macroeconomics: National Income Accounting, Simple Keynesian Model of Income Determination and the Multiplier, IS-LM Model, Model of Aggregate Demand and Aggregate Supply, Harrod-Domar and Solow Models of Growth, Money, Banking and Inflation.

Sample questions for ME II (Economics), 2010

1.

(a) Imagine a closed economy in which tax is imposed only on income. The government spending (G) is required (by a balanced budget amendment to the relevant law) to be equal to the tax revenue; thus $G = tY$, where t is the tax rate and Y is income. Consumption expenditure (C) is proportional to disposable income and investment (I) is exogenously given.

(i) Explain why government spending is endogenous in this model.

(ii) Is the multiplier in this model larger or smaller than in the case in which government spending is exogenous?

(iii) When t increases, does Y decrease, increase or stay the same? Give an answer with intuitive explanation.

(b) Consider the following macroeconomic model with notation having usual meanings: $C = 100 + 1.3Y$ (Consumption function), $I = \frac{500}{r}$ (Investment function), $M^D = 150Y + 100 - 1500r$ (Demand for money function) and $M^S = 2100$ (Supply of money). Do you think that there exists an equilibrium? Justify your answer using the IS-LM model.

[3+8+4]+[5]

2. Consider a market with two firms. Let the cost function of each firm be $C(q) = mq$ where $q \geq 0$. Let the inverse demand functions of firms 1 and 2 be $P_1(q_1, q_2) = a - q_1 - sq_2$ and $P_2(q_1, q_2) = a - q_2 - sq_1$, respectively. Assume that $0 < s < 1$ and $a > m > 0$.

(a) Find the Cournot equilibrium quantities of the two firms.

(b) Using the inverse demand functions $P_1(q_1, q_2)$ and $P_2(q_1, q_2)$, derive direct demand functions $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$ of firms 1 and 2.

(c) Using the direct demand functions $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$, find the Bertrand equilibrium prices.

[8]+[5]+[7]

3.

(a) A monopolist can sell his output in two geographically separated markets A and B. The total cost function is $TC = 5 + 3(Q_A + Q_B)$ where Q_A and Q_B are quantities sold in markets A and B respectively. The demand functions for the two markets are, respectively, $P_A = 15 - Q_A$ and $P_B = 25 - 2Q_B$. Calculate the firm's price, output, profit and the deadweight loss to the society if it can get involved in price discrimination.

(b) Suppose that you have the following information. Each month an airline sells 1500 business-class tickets at Rs. 200 per ticket and 6000 economy-class tickets at Rs. 80 per ticket. The airline treats business class and economy class as two separate markets. The airline knows the demand curves for the two markets and maximizes profit. It is also known that the demand curve of each of the two markets is linear and marginal cost associated with each ticket is Rs. 50.

(i) Use the above information to construct the demand curves for economy class and business class tickets.

(ii) What would be the equilibrium quantities and prices if the airline could not get involved in price discrimination?

[12]+**[4+4]**

4. Consider an economy producing two goods 1 and 2 using the following production functions: $X_1 = L_1^{\frac{1}{2}} K^{\frac{1}{2}}$ and $X_2 = L_2^{\frac{1}{2}} T^{\frac{1}{2}}$, where X_1 and X_2 are the outputs of good 1 and 2, respectively, K is capital used in production of good 1, T is land used in production of good 2 and L_1 and L_2 are amounts of labour used in production of good 1 and 2, respectively. Full employment of all factors is assumed implying the following: $K = \bar{K}, T = \bar{T}, L_1 + L_2 = \bar{L}$ where $\bar{K}$, $\bar{T}$ and $\bar{L}$ are total amounts of capital, land and labour available to the economy. Labour is assumed to be perfectly mobile between sectors 1 and 2. The underlying preference pattern of the economy generates the relative demand function, $\frac{D_1}{D_2} = \gamma \left(\frac{p_1}{p_2}\right)^{-2}$, where D_1 and D_2 are the demands and p_1 and p_2 prices of good 1 and 2 respectively. All markets (both commodities and factors) are competitive.

(a) Derive the relationship between $\frac{X_1}{X_2}$ and $\frac{p_1}{p_2}$.

(b) Suppose that γ goes up. What can you say about the new equilibrium relative price?

[15]+**[5]**

5. Consider the IS-LM representation of an economy with the following features:

(i) The economy is engaged in export and import of goods and services, but not in capital transactions with foreign countries.
(ii) Nominal exchange rate, that is, domestic currency per unit of foreign currency, e, is flexible.
(iii) Foreign price level (P^*) and domestic price level (P) are given exogenously.
(iv) There is no capital mobility and e has to be adjusted to balance trade in equilibrium. The trade balance (TB) equation (with an autonomous part $\bar{T} > 0$) is given by $TB = \bar{T} + \frac{\beta P^*}{P} - mY$, where Y is GDP and β and m are positive parameters, m being the marginal propensity to import.

(a) Taking into account trade balance equilibrium and commodity market equilibrium, derive the relationship between Y and the interest rate (r). Is it the same as in the IS curve for the closed economy? Explain. Draw also the LM curve on the (Y, r) plane.

(b) Suppose that the government spending is increased. Determine graphically the new equilibrium value of Y. How does the equilibrium value of e change?

(c) Suppose that P^* is increased. How does it affect the equilibrium values of Y and e?

[9]+**[6]**+**[5]**

6.

(a) A firm can produce its product with two alternative technologies given by $Y = \min\{\frac{K}{3}, \frac{L}{2}\}$ and $Y = \min\{\frac{K}{2}, \frac{L}{3}\}$. The factor markets are competitive and the marginal cost of production is $Rs.20$ with each of these two technologies. Find the equation of the expansion path of the firm if it uses a third production technology given by $Y = K^{\frac{2}{3}} L^{\frac{1}{3}}$.

(b) A utility maximizing consumer with a given money income consumes two commodities X and Y. He is a price taker in the market for X. For Y there are two alternatives: (A) He purchases Y from the market being a price taker, (B) The government supplies a fixed quantity of it through ration shops free of cost. Is the consumer necessarily better off in case (B)? Explain your answer with respect to the following cases:

(i) Indifference curves are strictly convex to the origin.
(ii) X and Y are perfect substitutes.
(iii) X and Y are perfect complements.

[14]+**[2+2+2]**

7. Indicate, with adequate explanations, whether each of the following statements is TRUE or FALSE.

(a) If an increase in the price of a good leads a consumer to buy more of it, then an increase in his income will lead him to buy less of the good. By the same argument, if an increase in the price of the good leads him to buy less of it, then an increase in his income will lead him to buy more of the good.

(b) Suppose that a farmer, who receives all his income from the sale of his crop at a price beyond his control, consumes more of the crop as a result of the price increase. Then the crop is a normal good.

(c) If the non-wage income of a person increases then he chooses to work less at a given wage rate. Then he will choose to work more as his wage rate increases.

(d) The amount of stipends which Indian Statistical Institute pays to its students is a part of GDP.

[8]+[4]+[4]+[4]

8. Consider a Solow model with the production function $Y = K^{\frac{1}{2}} L^{\frac{1}{2}}$, where Y, K and L are levels of output, capital and labour, respectively. Suppose, 20% of income is saved and invested. Assume that the rate of growth of labour force, that is, $\left(\frac{\frac{dL}{dt}}{L}\right) = 0.05$.

(a) Find the capital-labour ratio, rate of growth of output, rate of growth of savings and the wage rate, in the steady state growth equilibrium.

(b) Suppose that the proportion of income saved goes up from 20% to 40%. What will be the new steady state growth rate of output?

(c) Is the rate of growth of output in the new steady state equilibrium different from that obtained just before attaining the new steady state (after deviating from the old steady state)? Explain.

[8]+[4]+[8]

9.

(a) Consider the utility function $U(x_1, x_2) = (x_1 - s_1)^{0.5}(x_2 - s_2)^{0.5}$, where $s_1 > 0$ and $s_2 > 0$ represent subsistence consumption and $x_1 \geq s_1$ and $x_2 \geq s_2$. Using the standard budget constraint, derive the budget share functions and demand functions of the utility maximizing consumer. Are they linear in prices? Justify your answer.

(b) Suppose that a consumer maximizes $U(x_1, x_2)$ subject to the budget constraint $p(x_1 + x_2) \leq M$ where $x_1 \geq 0$, $x_2 \geq 0$, $M > 0$ and $p > 0$. Moreover, assume that the utility function is symmetric, that is $U(x_1, x_2) = U(x_2, x_1)$ for all $x_1 \geq 0$ and $x_2 \geq 0$. If the solution (x_1^*, x_2^*) to the consumer's constrained optimization problem exists and is unique, then show that $x_1^* = x_2^*$.

[10]+[10]

10.

(a) Consider an economy with two persons (A and B) and two goods (1 and 2). Utility functions of the two persons are given by $U_A(x_{A1}, x_{A2}) = x_{A1}^{\alpha} + x_{A2}^{\alpha}$ with $0 < \alpha < 1$; and $U_B(x_{B1}, x_{B2}) = x_{B1} + x_{B2}$. Derive the equation of the contract curve and mention its properties.

(b) (i) A firm can produce a product at a constant average (marginal) cost of Rs. 4. The demand for the good is given by $x = 100 - 10p$. Assume that the firm owner requires a profit of Rs. 80. Determine the level of output and the price that yields maximum revenue if this profit constraint is to be fulfilled.

(ii) What will be the effects on price and output if the targeted profit is increased to Rs. 100?

(iii) Also find out the effects of the increase in marginal cost from Rs. 4 to Rs. 8 on price and output.

[8]+[6+3+3]